PMM U.S.S.R.,Vol.45,pp.69-76
0021-8928/82/1/0069 $7.50/0

UDC 533.6

HIGH-FREQUENCY ASYMPTOTICS OF SOLUTION OF THE NONLINEAR PROBLEM NEAR A CAUSTIC*

S. M. MANUKIAN

The determination of parameters of motion of nonlinear dispersing inhomogeneous media is considered in the case of quasi-simple harmonic small amplitude waves near a caustic. A nonlinear equation is obtained for high-frequency asymptotics of wave amplitude in an arbitrary weakly nonlinear dispersing nonhomogeneous medium, which is in essence an ordinary second order differential equation which in the linear case becomes the Airy equation. This confirms the validity of standard solutions /1/ for dispersing linear media and simple harmonic waves.

Equations are derived here on the assumption of smooth variation of wave parameters, which is possible for media with cubic nonlinearity or for those with weak quadratic nonlinearity with strong dispersion. Unlike in the case of media with quadratic nonlinearity in which the equation of short waves is valid near a caustic /2 – 5/, an ordinary nonlinear differential equation whose solution has different properties for stable (defocusing) and unstable (focusing) media, is obtained for smooth quasi-simple harmonic waves in cubic media.

The Airy function was used in /1/ for deriving a solution near a caustic of an arbitrary linear hyperbolic system with variable coefficients for a time periodic wave. A linear solution was obtained in /6 – 8/ for unsteady simple harmonic waves near the caustic. That solution at the caustic has a singularity whose elimination necessitates obtaining simplified nonlinear equations of short waves and finding their solutions /2/. Equations near a caustic in nonlinear formulation for unsteady low intensity waves were obtained in /3,4,9/. A numerical solution of the nonlinear problem appears in /5,10,11/. Equations for periodic shock waves and methods of their solution are given in /12/. Investigation of waves, close to simple harmonic, near a caustic when variation of wave amplitude and phase is smooth prior discontinuity formation is also of interest. General modulation equations for such waves were obtained in /13—16/. In deriving the modulation equation it is possible to assume, as in geometric optics, the basic wave frequency to be high /15 and 16/, which is equivalent to the assumption of slow variation of amplitude and phase.

1. Derivation of nonlinear equations.

Let us obtain the equations for slow variation of amplitudes on phases of a quasi-simple harmonic wave near a caustic. The obtained in /1/ linear solution near a caustic depends on two variables /1,3,6—8/

$$x^* = (\mathbf{x} - \mathbf{x}^\circ)\mathbf{k}, \quad y = (\mathbf{x} - \mathbf{x}^\circ)\mathbf{N} \tag{1.1}$$

where $\mathbf{k} = \{\alpha_j\}$, $j = 1, 2, 3, \ldots, n$ is the vector of the normal to the wave at point A of ray tangency to the caustic surface, N is the unit vector of the normal to the caustic at point A directed toward its concavity, y is the distance of point $\mathbf{x}$ from the caustic, x^* is the time of wave run along the ray from A to point $\mathbf{x}$, and $\mathbf{x}^\circ$ is the radius vector of point A.

The solution implies the following orders of smallness of parameters:

$$x^* \sim \varepsilon^{3/2}, \quad y \sim \varepsilon \tag{1.2}$$

where for a step-wave $\varepsilon \sim \gamma^{4/3}$ and γ is the wave intensity away from the caustic.

As in /1,13/, we consider a high-frequency asymptotics of the problem. Let in a linear formulation the medium be defined by the equation

$$\Delta(ip_t, -ip_j, \mathbf{x})\Phi = 0, \quad p_t = \frac{\partial}{\partial t} = \frac{\partial}{\partial x^*}\frac{\partial x^*}{\partial t} + \frac{\partial}{\partial y}\frac{\partial y}{\partial t}, \quad p_j = \alpha_j\frac{\partial}{\partial x^*} + N_j\frac{\partial}{\partial y} \tag{1.3}$$

where Δ is some linear operator with variable coefficients of the form of a polynomial of n-th power.

In accordance with (1.1) the second term in p_t is unessential. Expanding Δ in powers of small operators $N_j\partial/\partial y$, $(1 + \partial x^*/\partial t)\partial/\partial x^*$ and retaining only higher derivatives, we obtain

*Prikl.Matem.Mekhan.,45,No.1,95-104,1981

$$\Delta(ip_t - ip_j, \mathbf{x}) = \Delta(-i\partial/\partial x^*, -i\alpha_j\partial/\partial x^*, \mathbf{x}^\circ)\Phi + L_0 + L_{1x} + L_{2x} + L_{1y} + L_{2y} \tag{1.4}$$

$$L_0 = \frac{\partial\Delta}{\partial x_k}(x_k - x_k^\circ), \quad L_{1x} = i\Delta_{-i\partial/\partial x^*}(x_k - x_k^\circ)\frac{\partial\alpha_k}{\partial t}\frac{\partial}{\partial x^*}$$

$$L_{2x} = \Delta_{-\partial^2,\partial x^{*2}}\left[(x_k - x_k^\circ)\frac{\partial\alpha_k}{\partial t}\right]^2\frac{\partial^2}{\partial x^{*2}}$$

$$L_{1y} = \left(-\Delta_{-i\alpha_j\partial/\partial x^*} + \Delta_{(-i\partial/\partial x^*)(-i\alpha_j/\partial x^*)}\frac{\partial\alpha_k}{\partial t}\right)N_j\frac{\partial}{\partial y}$$

$$L_{2y} = -\frac{1}{2}\Delta_{-\alpha_i\alpha_j\partial^2/\partial x^{*2}}N_iN_j\frac{\partial^2}{\partial y^2}$$

where the subscript at the operator Δ denotes differentiation, for instance $\Delta_{-i\partial/\partial x^*} = \partial\Delta/\partial(-i\partial/\partial x^*)$.

Let us set in conformity with /15/ $\Phi = \psi e^{i\omega x^*}$ which is in agreement with the linear solution near a caustic and where ω is the unperturbed wave frequency. From this follows that $\Delta \sim \omega^n$, $\Delta_\omega \sim \omega^{n-1}$, $\omega \sim \varepsilon^{-3/2}$ /15/. We omit in L_{2x} and the second term in L_{1y} since they are of a higher order of smallness $\varepsilon^{2-3n/2}$ than the remaining terms of order $\varepsilon^{1-3n/2}$. We also omit in (1.4) terms of order $\varepsilon^{3(1-n)/2}$ obtained by the action of operators on variable coefficients. Taking into account the dispersion relation at point A, the equality $\Delta(\omega, \alpha_j, \mathbf{x}^\circ) = 0$, and the relation $N_j\Delta_{\alpha_j} = 0$, we obtain for a slowly varying amplitude ψ the expression /15/

$$e^{-i\omega x^*}\Delta\Phi = -\psi\omega\Delta_\omega(x_j - x_j^\circ)\left(\frac{\partial\alpha_j^1}{\partial t} - \frac{\partial\alpha_j}{\partial t}\right) - \frac{1}{2}\Delta_{\alpha_i^*\alpha_j^*}N_iN_j\frac{\partial^2\psi}{\partial y_1^2} \tag{1.5}$$

where the ray equation is used

$$\frac{dx_i}{ds} = \alpha_i^*, \quad \frac{dt}{ds} = -\Delta_\omega, \quad \frac{d\alpha_j^*}{ds} = -\Delta_{x_j}, \quad \alpha_j^* = \alpha_j\omega, \quad \alpha_j = \frac{\partial x^*}{\partial x_j} \tag{1.6}$$

On the basis of (1.1) we set $x_i - x_i^0 = y_1N_i - y_1\alpha_jN_j\Delta_{\alpha_i}/(\alpha_k\Delta_{\alpha_k})$. Using the notation

$$\lambda_1 y_1 = -\omega\Delta_\omega(x_j - x_j^\circ)\left(\frac{\partial\alpha_j^1}{\partial t} - \frac{\partial\alpha_j}{\partial t}\right)$$

we obtain

$$\lambda_1 = -\omega\Delta_\omega\left(N_j - \frac{\alpha_iN_i}{\alpha_k\Delta_{\alpha_k}}\Delta_{\alpha_j}\right)\left(\frac{\partial\alpha_i^1}{\partial t} - \frac{\partial\alpha_j}{\partial t}\right)$$

from which and (1.5) we have

$$d^2\psi/dy_1^2 - \varkappa y_1\psi = 0, \quad \varkappa = 2\lambda_1/(\Delta_{\alpha_i^*\alpha_j^*}N_iN_j) \tag{1.7}$$

The solution of Eq. (1.7) near a caustic expressed in terms of Airy's function $v(y)$ /12/ is of the form

$$\psi = Kv(y_*)\exp\left[i\pi(k/2 + 3/4)\right], \quad y_* = y_1\varkappa^{1/3}, K = B\omega \tag{1.8}$$

where the constant B which defines the ray solution away from the caustic is

$$\psi_\infty = (-i\omega)^{-k-1}B(-y_1)^{-1/4}\exp\left[{}^2/_3 i(-y_1)^{3/2}\right]$$

For deriving the nonlinear equation which defines the slow amplitude variation in conformity with the method used in the theory of modulation /13/, we apply the nonlinear dispersion formula obtained by varying the Lagrangian

$$L^\circ = \int L\,dx\,dt$$

averaged over the amplitude. It should be noted that taking into account the quadratic nonlinearity near a caustic in unsteady nonlinear problems as was made in /9,10,15/ for slowly varying amplitudes in a nondispersing medium, does not result in nonzero additions in (1.7). In the case of a dispersing medium with quadratic nonlinearity, for example of the coefficient of dynamic viscosity (KDV) in the nonlinear dispersion formula, a nonzero nonlinear term is generated /15/. To take the latter into account we write /13,15/ the nonlinear dispersion formula which is valid in the slow modulation region, i.e. away from a caustic, as

$$\omega \approx \omega_0 (k, x) + (\partial\omega/\partial a^2)_{a=0} a^2 \tag{1.9}$$
$$\omega \approx -\omega_{00}\partial (x^* + \varphi)/\partial t, \quad \omega_\infty (x^* + \varphi) = F$$
$$\omega_\infty = \omega_0 (k^\circ, x^\circ), \quad \varphi = \varphi (y_1)$$
$$\omega_0 \approx \omega_{00} + \frac{\partial\omega_0}{\partial x_k}(x_k - x_k{}^\circ) + \frac{\partial\omega_0}{\partial \mathbf{k}}(\mathbf{k} - \mathbf{k}^\circ) + \frac{1}{2}\frac{\partial^2\omega_0}{\partial\alpha_i\partial\alpha_j}(\alpha_j - \alpha_j{}^1)(\alpha_i - \alpha_i{}^1)$$

where F is the eikonal, and $\mathbf{k}^0 = \{\alpha_j{}^1\}$ is the value of vector $\mathbf{k}$ at point A. The constant ω_∞ is the same as ω in (1.5).

Using (1.1), the definition of $\alpha_j{}^*$ in (1.6), the dispersion equation $\Delta = \omega - \omega_0 (\alpha_j, x_j)$ for the nonlinear problem, and Eq. (1.6) for rays, we obtain

$$\lambda_1 y_1 + \frac{1}{2}\Delta_{\alpha_i\alpha_j} N_i N_j \left(\frac{\partial\varphi}{\partial y_1}\right)^2 - \left(\frac{\partial\omega}{\partial a^2}\right)_{a=0} a^2 = 0 \tag{1.10}$$

which is the nonlinear equation for the perturbed phase φ.

Let us assume that unlike in the linear solution ψ also depends on x^*. Then the application in (1.4) of the first operator to the product $\psi e^{i\omega x^*}$ in its right-hand side, retaining only the first derivatives with respect to ψ, yields

$$-\frac{\omega\Delta_\omega + \alpha_j\Delta_{\alpha_j}}{\omega}\, i\,\frac{\partial\psi}{\partial x^*}\, e^{i\omega x^*} \tag{1.11}$$

The right-hand side of (1.5) is then supplemented by the term $i\omega^{-1}(\omega\Delta_\omega + \alpha_j\Delta_{\alpha_j}) \times \partial\psi/\partial x^*$, and if (1.11) is also to be of order $\varepsilon^{1-3n/2}$, it is necessary to set $\partial\psi/\partial x^* \sim \psi\omega^{1/3}$. In the case of nondispersing media for which Δ is a homogeneous function that term is zero. This occurs, for instance, for a conducting medium considered below. In a linear problem the addition of this term corresponds to the substitution for Airy's function of the function

$$v = \mathrm{Re}\left[\frac{2}{\sqrt{\pi}}\int_\eta^\infty \exp\left[i\left(\frac{\tau^3}{3} + y_*\tau\right)\right]d\tau\right], \quad \eta = -\frac{\omega\lambda_1}{\varkappa^{1/3}(\omega\Delta_\omega + \alpha_j\Delta_{\alpha_j})}\, x^*$$

which satisfies the equation

$$\partial^2 v/\partial y_*{}^2 - y_* v - i\partial v/d\eta = 0$$

Since $x^* \sim 1/\omega$, $y_* \sim 1$, hence $\tau \sim 1$, and we assume that $\eta \approx 0$, which corresponds to the rejection of the term (1.11).

We set in (1.5) $\psi = ae^{i\varphi}$, equate the right-hand side to zero, and separate the real part of the obtained equation. Thus:

$$\frac{1}{2}\Delta_{\alpha_i\alpha_j} N_i N_j \left[a\left(\frac{\partial\varphi}{\partial y_1}\right)^2 - \frac{\partial^2 a}{\partial y_1{}^2}\right] + a\lambda_1 y_1 = 0$$

Omitting the diffraction term $\partial^2 a/\partial y_1{}^2$ and comparing with Eq. (1.10), we obtain near the caustic a nonlinear equation of the form

$$\frac{1}{2}\Delta_{\alpha_i{}^*\alpha_j{}^*} N_i N_j \frac{\partial^2\psi}{\partial y_1{}^2} - \lambda_1 y_1 \psi + \Delta_\omega \left(\frac{\partial\omega}{\partial a^2}\right)_{a=0} \psi |\psi|^2 = 0 \tag{1.12}$$

For a defocusing medium $\partial\omega/\partial a^2 > 0$, hence near the caustic Eq. (1.12) assumes the form

$$\partial^2\psi_1/\partial y_*{}^2 - y_*\psi_1 - \psi_1 |\psi_1|^2 = 0 \tag{1.13}$$
$$\psi = \psi_1\mu \exp [i\pi (k/2 + {}^3/_4)]$$
$$\mu = \{-\lambda_1\nu/[\Delta_\omega(\partial\omega/\partial a^2)_{a=0}]\}^{1/2}$$
$$\nu = \varkappa^{-1/3}, \quad \lambda_1 < 0, \quad \Delta_\omega > 0$$

Since in the linear problem by virtue of (1.8) ψ_1 is real, we set in the nonlinear problem $|\psi_1|^2 = \psi_1{}^2$. The linear asymptotics for ψ_1 is of the form

$$\psi_1 = K\mu^{-1} v (y_*) \tag{1.14}$$

Away from the caustic we seek a solution of Eq. (1.13) by the method of slowly varying amplitudes and phases; after separating real and imaginary parts, we obtain equations

$$a\left(\frac{d\varphi}{dy_*}\right)^2-\frac{\partial^2 a}{\partial y_*^2}+y_*a+a^3=0,\quad a\frac{d^2\varphi}{dy_*^2}+2\frac{\partial a}{\partial y_*}\frac{\partial\varphi}{\partial y_*}=0 \tag{1.15}$$

Rejection of the diffraction term $\partial^2a/\partial y_*^2$ yields the nonlinear dispersion formulas; further rejection of the nonlinear term a^3 yields an equation whose solution for large in absolute value negative y_* is given by asymptotics (1.8). Coefficients of Eqs.(1.12) depend on specific media.

1°. Let us determine the coefficients of equations of electrodynamics of slender beams. We write the equation for the electric field intensity E on the assumption that $\varepsilon_2a^2 \ll \varepsilon_0$ as /13/

$$\nabla^2\mathbf{E}+\nabla\left(\frac{\nabla\varepsilon_0}{\varepsilon_0}\mathbf{E}\right)+k^2(1+\varepsilon_2a^2)\,\mathbf{E}=0 \tag{1.16}$$

whose solution for a monochromatic wave is of the form

$$\mathbf{E}=(a\mathbf{e}_0\exp(-i\tau)+a\mathbf{e}_0\exp(i\tau)),\quad k=\omega\sqrt{\varepsilon_0}/c,\ \partial\tau/\partial t=-1,\quad \partial\tau/\partial x_i=\alpha_1^*,\ \mathbf{E}=(\mathbf{E}_1+\mathbf{E}_2)/2$$

where $\mathbf{e}_0$ is the unit vector of linear polarization, ∇ is the Hamiltonian $a=|\mathbf{E}_1|$, c is the speed of light, τ is the eikonal, ε_0 and $\varepsilon_0(1+\varepsilon_2|\mathbf{E}_1|^2)$ are, the refraction indices in the linear and nonlinear problems, respectively. We select a linear dispersion formula of the form

$$\Delta=\omega_0-c\left(\sum_{j=1}^{3}\alpha_j^{*2}\right)^{1/2}\varepsilon_0^{-1/2}$$

Then

$$\Delta_{\alpha_j^*}=-c\varepsilon_0^{-1/2}\alpha_j^*/\alpha,\quad \Delta_{\alpha_i^*\alpha_j^*}N_iN_j=-c^2/(\varepsilon_0\omega_0)$$
$$\partial\omega/\partial a^2=-{}^1/_2\varepsilon_2\omega_0<0,\quad \lambda_1=-\omega/R$$
$$R^{-1}=R_s^{-1}-R_r^{-1},\quad \alpha=(\alpha_1^2+\alpha_2^2+\alpha_3^2)^{1/2}$$

where R_r^{-1} is the projection of the ray curvature vector on the normal to the caustic, R_s^{-1} is the projection of the vector of curvature of the respective surface ray on the normal to the caustic. Equation (1.12) assumes the form

$$\frac{c^2}{2\varepsilon_0\omega_0}\frac{d^2\psi}{dy_1^2}-\frac{\omega_0}{R}y_1\psi+\frac{\omega_0\varepsilon_0}{2}\psi|\psi|^2=0 \tag{1.17}$$

which relates to a focusing medium in which $(\partial\omega/\partial a^2)_{a=0}<0$. Then, instead of (1.13) we have the equation

$$\frac{d^2\psi_1}{dy_*^2}-y_*\psi_1+\psi_1^3=0,\quad \mu=\{\lambda_1\nu/[\Delta_\omega(\partial\omega/\partial a^2)_{a=0}]\}^{1/2} \tag{1.18}$$

2°. For waves on the surface of a deep wave we obtain a nonlinear dispersion relation of the form /14/

$$\omega^2=g\alpha^*(1+\alpha^{*2}a^2),\quad \alpha^*=(\alpha_1^2+\alpha_2^2)^{1/2}$$

where a is the fluid surface elevation amplitude and g is the acceleration of gravity. The coefficients of Eq.(1.12) can be specified similarly. The linear dispersion equation has the form $\Delta=\omega_0-\sqrt{g\alpha^*}$ and then

$$\Delta_{\alpha_j^*}=-\frac{g^{1/2}\alpha_1^*}{2\alpha^{*1/2}},\quad \Delta_{\alpha_i^*\alpha_j^*}N_iN_j=-\frac{g^{1/2}}{2\alpha^{*3/2}},\quad \lambda_1=-\frac{\omega_0}{2R},\quad \frac{\partial\omega}{\partial a^2}=\frac{1}{2}\omega_0\alpha^{*2}$$

and Eq.(1.12) can now be written as

$$\frac{g^{1/2}}{4\alpha^{*3/2}}\frac{d^2\psi}{dy_1^2}-\frac{\omega_0}{2R}y_1\psi-\frac{1}{2}\alpha^{*2}\omega_0\psi|\psi|^2=0 \tag{1.19}$$

Thus Eq.(1.19) relates to a defocusing medium and reduces to the form (1.13).

3°. For waves on the surface of water of finite depth h_0 the nonlinear dispersion relation is of the form /13/

$$\omega=\omega_0(k)+\Omega_2(k)\frac{k^2ga^2}{2c_0},\quad \Omega_2(k)=\frac{9T_0^4-10T_0^2+9}{8T_0^3}-\frac{1}{kh_0}\left\{\frac{(2C_0-c_0/2)^2}{gh_0-C_0^2}+1\right\} \tag{1.20}$$

$$T_0 = \operatorname{th}(kh_0), \quad \omega_0^2(k) = gk \operatorname{th}(kh_0), \quad C_0 = \omega_0'(k), \quad c_0 = \frac{\omega_0}{k}$$

As shown in /13/, $\omega_0' < 0, \Omega_2 > 0$ when $kh_0 > 1{,}36$ and $\Omega_2 < 0$ when $kh_0 < 1{,}36$. Thus for waves over deep water $(\partial\omega / \partial a^2)_{a=0} > 0$ and there is transverse stability of the wave /16/. The medium is defocusing relative to transverse perturbations, i.e. near a caustic the equation reduces to (1.13), while for shallow water $(\partial\omega / \partial a^2)_{a=0} < 0$, the medium has focusing properties, and Eq.(1.18) holds.

4°. Let us determine the nonlinear dispersion relation for waves on the surface of infinitely deep water covered by a thin elastic plate. For simplicity we take into account only the physical nonlinearity defined by the nonlinear shear modulus $\gamma_2 G$, where G is the linear shear modulus /17/. The Lagrangian L is represented in the form of sum $L_0 + L_1$, where L_0 and L_1 are, respectively, the Lagrangians for water and plate. Representing, as in /13/, water elevation in the form of quasi-simple harmonic waves $h' = a\cos\tau + b\cos\tau$, introducing the averaged Lagrangian

$$L^* = \frac{1}{2\pi}\int_0^{2\pi} L\,d\tau$$

and varying L^* with respect to a, as in /18/, we obtain

$$\omega_0\left(\frac{\partial\omega}{\partial a^2}\right)_{a=0} = \frac{T}{\rho h + \rho' k^{-1}} \tag{1.21}$$

$$T = \frac{\gamma_2 G h^6 k^8 (1-\nu+\nu^2)^2}{180(1-\nu)^4} + \frac{1}{4}\omega_0^2\rho' k\left(3 - \frac{2\varkappa}{k}\right)$$

$$\omega_0^2(k) = \frac{6g\rho'(1-\nu) + Gh^3k^4}{6(1-\nu)(\rho h + \rho' k^{-1})}$$

$$\varkappa = \frac{\rho^1\omega_0^2(k)}{2g\rho' - 8h\rho\omega_0^2(k) + 166h^3k^4(1-\nu)^{-1}/3}$$

where ω_0 is the frequency in the linear problem, ρ' and ρ are densities of water and plate, respectively, h is the plate thickness, and ν is the Poisson's ratio.

When $\rho' = 0$, we have for the plate the nonlinear dispersion equation

$$\omega_0\left(\frac{\partial\omega}{\partial a^2}\right)_{a=0} = \frac{\gamma_2 G h^4 k^8 (1-\nu+\nu^2)^2}{180\rho(1-\nu)^4}, \quad \omega_0^2 = \frac{Gh^2k^4}{6(1-\nu)\rho} \tag{1.22}$$

For an incompressible plate for which $\nu = 1/2$ formula (1.22) was obtained in /18/. Since for metal plates $\gamma_2 < 0$ /17/, hence $(\partial\omega / \partial a^2)_{a=0} < 0$, i.e. the medium has focusing properties /18/. It is interesting to investigate the dependence of $(\partial\omega / \partial a^2)_{a=0}$ on the values of $\xi = G/(hg\rho'), \gamma_2, kh$. Computations were carried out for fixed $\gamma_2 = 10^6, \xi = 10^6, \xi = 10^4$ and $\gamma_2 = -10^6, \xi = 10^6, \xi = 10^4$ for kh varied from 0 to 0.5. In the first variant with $\gamma_2 = -10^6, \xi = 10^6$, $(\partial\omega / \partial a^2) < 0$ was obtained everywhere, except for $kh = 0$, where $(\partial\omega / \partial a^2)_{a=0} = 0{,}25$. In the second variant $(\partial\omega / \partial a^2)_{a=0}$ changed its sign when kh was increased. The dependence of $\eta = (\partial\omega / \partial a^2)_{a=0}\omega_0(gk^2)^{-1}$ on kh is shown in Fig.1. It will be seen that for $0 < kh < 0{,}21$ the medium has defocusing properties relative to transverse oscillations and focusing ones when $kh \geqslant 0{,}21$. It should be pointed out that formula $\psi = ae^{i\varphi}$ has any meaning only away from a caustic, where the incident and the reflected waves separate, while near a caustic Eqs.(1.13) and (1.18) are to be solved by linking for large $|y_*|$ the values of ψ_1 with the linear asymptotic (1.8).

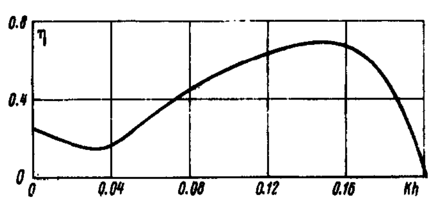

Fig.1

2. Statement and solution of the boundary value problem. Let us find the solution of Eq.(1.18) or (1.13) when $|\psi_1|^2 = \psi_1^2$ near a caustic for a given asymptotic (1.8). Equations (1.13) and (1.18) are equations of Painlevé and have two mobile singular points /19/. Hence if the Cauchy condition is specified at point $y_* = -5$, which we assume fairly distant from a caustic, the problem of initial conditions has no continuous solution. Numerical computation shows that the solution of the Cauchy problem rapidly approaches ∞. The boundary value problem for segment $[-5, 5]$ can be taken as an approximate substitution for the problem stated at the beginning of this Section. At the end of that segment ψ_1 is defined in conformity with the linear solution (1.8). The problem for the second order differential equation is thus solved.

$$\psi_1'' = y\psi_1 \pm \psi_1^3,\ \ \psi_1^- = Kv^-/\mu,\ \ \psi_1^+ = Kv^+/\mu,\quad f^- = f(-5),\ f^+ = f(+5) \tag{2.1}$$

where $v(y)$ is the Airy function and $K = \text{const}$. By this we also stipulate that the solution of Eq.(2.1) must merge at boundaries with the linear solution. To obtain homogeneous boundary conditions we use the notation $\psi^* = \psi_1 - Kv(y)/\mu$ with which the input problem (2.1) assumes the form

$$\psi^{*\prime\prime} = \psi^* y \pm (\psi^* + Kv/\mu)^3,\ \ \psi^{*-} = \psi^{*+} = 0 \tag{2.2}$$

whose solution we seek in the form

$$\psi^+ = \pm \int_{-5}^{5} G(y,\xi)\left[\psi^*(\xi) + \frac{Kv(\xi)}{\mu}\right]^3 d\xi \tag{2.3}$$

where $G(y,\xi)$ is the Green's function that satisfies the equation $G'' = yG + \delta(y-\xi)$ in which δ denotes the delta function. We split function G into two parts

$$-5 < y < \xi,\ G(y,\xi) = \psi_1^*(y,\xi)$$
$$\xi < y < 5,\ G(y,\xi) = \psi_2^*(y,\xi)$$

where $\psi_{1,2}^*$ are solutions of the linear problem. We obtain these by the method of indeterminate coefficients

$$\psi_1^* = c_1 v(y) + c_1^* u(y),\ \psi_2^* = c_2 v(y) + c_2^* u(y)$$

Relations

$$\psi_1^*(\xi,\xi) = \psi_2^*(\xi,\xi),\ \psi_2^{*\prime}(\xi,\xi) - \psi_1^{*\prime}(\xi,\xi) = 1 \tag{2.4}$$

are valid for functions ψ_1^*, ψ_2^*.

Using formulas (2.4) and the boundary conditions, we obtain for coefficients c_1, c_1^*, c_2, c_2^* the following expressions

$$c_1 = d^{-1}[-p_2u^+u^- + p_2v^+u^-],\ c_1^* = d^{-1}[p_2u^+v^- - p_1v^+v^-]$$
$$c_2 = d^{-1}[-p_2u^+u^- + p_1u^+v^-],\ c_2^* = d^{-1}[p_2u^-v^+ - p_1v^+v^-]$$
$$d = u^+v^- + u^-v^+,\ p_1 = u(\xi)/w(\xi),\ p_2 = v(\xi)/w(\xi)$$
$$w(\xi) = u(\xi)v'(\xi) - v(\xi)u'(\xi)$$

Function (2.3) satisfies Eq.(2.2). The solution of the differential equations thus reduces to the solution of the following integral equation:

$$\psi^* = \pm \int_{-5}^{y} \psi_2^*(y,\xi)\left(\psi^* + \frac{Kv}{\mu}\right)^3 d\xi \pm \int_{y}^{5} \psi_1^*(y,\xi)\left(\psi^* + \frac{Kv}{\mu}\right)^3 d\xi \tag{2.5}$$

which is solved by the improved method of successive approximations /20/ with the linear

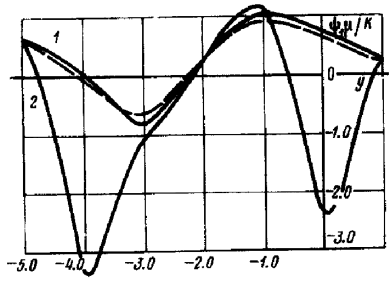

Fig.2

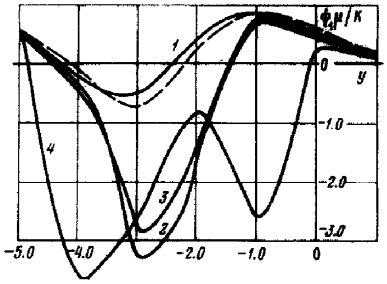

Fig.3

solution taken as the zero approximation. Computation results for a focusing medium are shown in Fig.2, where curve *1* corresponds to solution $\psi_1/(K/\mu)$ for $K/\mu = 0{,}4$ and curve *2* for $K/\mu = 0{,}5$. Curves *1–4* in Fig.3 were calculated for a medium with defocusing properties. These curves correspond to $K/\mu = 0{,}4;\ 0{,}5;\ 0{,}7;\ 1{,}0$, respectively. The linear solution is shown in Figs.2 and 3 by dash lines. Note the good agreement with the nonlinear solution for $y = \pm 5$.

These results show that the nonlinearity introduces marked changes in the intensity distribution along the normal to the caustic at transition form the region of light to that of shadow. In the linear problem these changes are fairly smooth. An abrupt change of solution on the wave of the "soletone character" occurs in the nonlinear problem /16/, and the bursts in the case of a defocusing medium (Fig.3) are smaller than in the case of focusing one (Fig.2).

3. On the absence of branching of the boundary value problem solution.

Let us prove the uniqueness of the solution of the boundary value problem (2.1) when ε is fairly small. Using the notation $\psi^* = K\varphi/\mu$, $\varepsilon = \pm (K/\mu)^2$ we reduce (2.2) to the form

$$\varphi'' = y\varphi + \varepsilon(\varphi + v)^3, \quad \varphi|_{y=c} = \varphi|_{y=b} = 0 \tag{3.1}$$

In investigating the uniqueness of solution we use the results of /12/ according to which it is necessary to consider for Eq.(3.1) the Cauchy problem

$$\varphi'' = y\varphi + \varepsilon(\varphi + v)^3, \quad \varphi|_{y=c} = 0, \ \varphi'|_{y=c} = a \tag{3.2}$$

where a is a parameter determined by the second of conditions (3.1). We seek a solution of this problem of the form

$$\varphi = c_1(y)v(y) + c_2(y)u(y)$$

From (3.2) we have

$$\varphi = \left[\int_c^y \frac{\varepsilon u U^3 d\xi}{uv' - vu'} + c_1^*\right] v(y) + \left[-\int_c^y \frac{\varepsilon v U^3 d\xi}{uv' - vu'} + c_2^*\right] u(y)$$

$$U = v + c_1 v + c_2 u, \quad c_1^* = -c_2^* = \frac{au(c)}{u(c)v'(c) - v(c)u'(c)}$$

where the boundary condition at $y = c$ was used for determining c_1^* and c_2^*. Using the second boundary condition we obtain

$$\left[\int_c^b \frac{\varepsilon u U^3 d\xi}{uv' - vu'} + c_1\right] v(b) + \left[-\int_c^b \frac{\varepsilon v U^3 d\xi}{uv' - vu'} + c_2\right] u(b) = 0 \tag{3.3}$$

According to /21/ problem (3.1) has a unique solution, when parameter a in (3.3) is single-valued. From (3.3) we have for a

$$a = \frac{(A_{uv} - A_{vu})D(c)}{B}, \quad A_{uv} = u(b)\int_c^b \frac{\varepsilon v U^3 d\xi}{D(\xi)} \tag{3.4}$$

$$D(\xi) = uv' - vu', \ B = u(c)v(b) - v(c)u(b)$$

If ε is fairly small, then $c_1 \sim c_1^* \sim c_2 \sim c_2^* \sim a \sim \varepsilon$ and rejecting in U smalls of order ε we assume U to be v in the right-hand side of (3.4). Thus a is uniquely determined, and by virtue of Theorem 1 in /21/ the problem has a unique solution.

The author thanks A. G. Bogdanov for his assistance.

REFERENCES

1. LUDWIG D., Uniform asymptotic expansions at a caustic. Communs. Pure and Appl. Math., Vol.19, No.6, p.216, 1966.
2. GAZARIAN Iu.L., On propagation of sound in inhomogeneous media. In: Problems of the Dynamic Theory of Seismic Wave Propagation, No.5, Izd. LGU, p.73, 1961.
3. BABICH V.M., The analytical properties of unsteady wave field in the neighborhood of a caustic. In: Problems of the Dynamic Theory of Seismic Wave Propagation., No.5. Izd. LGU, p.115, 1961.
4. BAGDOEV A.G. and OGANIAN G.G., Determination of gas parameters near a caustic. Dokl. Akad. Nauk ArmSSR, 49, No.2, p.69, 1969.
5. RYZHOV O.S. and KHRISTIANOVICH S.A., On nonlinear reflection of weak shock waves. PMM, Vol.22, No.5, 1958.

6. BAGDOEV A.G., Determination of parameters of motion of a medium near a caustic. Izv. Akad. Nauk ArmSSR, Mekhanika, Vol.23,No.2, 1970.

7. BAGDOEV A.G., Certain Nonlinear Problems of Compressible Fluid Motion. Erevan. Izd. Akad. Nauk ArmSSR, 1967.

8. GNIRAND J.P., Le caustique geometrique et la focalisation. Comp. Rend. Acad. Sci. Colon. Vol.260, No.6, 1965.

9. SCELASS R., MURMAN E.M., and KRUPP J.A., Finite difference calculation of the behavior of a discontinuous signal near a caustic. 3-rd Conf. on Sonic Boom Research, Washington, 1970. Washington, NASA, 1971.

10. EREMENKO V.A. and RYZHOV O.S., On the structure of a stream in the neighborhood of shock wave intersection with a caustic. Dokl. Akad. Nauk SSSR, Vol.238, No.3, 1978.

11. MANUKIAN S.M., On numerical calculation of the neighborhood of a caustic. Uch. Zap. Erevan. Univ., No.1, p.12, 1976.

12. MANUKIAN S.M., Numerical calculation of the nonlinear problem of periodic shock wave propagation near a caustic. Uch. Zap. Erevan. Univ, No.2, p.16, 1976.

13. WHITHAM G.B., Linear and Nonlinear Waves. Moscow, MIR, 1977.

14. LITVAK A.G., FRAIMAN G.M., and IUNAKOVSKII A.D., On the self-focusing axisymmetric beams of electromagnetic waves. Tr. Vses. Simpoziuma po Teorii Difraktsii i Rasprostraneniiu Voln. Tsakhadzor, Vol.2, Erevan, 1973.

15. BAGDOEV A.G., Determination of wave front neighborhood in the three-dimensional problem. Izv. Akad. Nauk ArmSSR. Mekhanika, Vol.30, No.6, 1977.

16. KARPMAN, V.I., Nonlinear Waves in Dispersive Media. English translation, Pergamon Press, 1975.

17. BAGDOEV A.G. and MOVSISIAN L.A., On the problem of flexural wave propagation in nonlinearly elastic plates. Izv. Akad. Nauk ArmSSR. Mekhan., Vol.32, No.5, 1979.

18. KAUDERER K., Nonlinear Mechanics. Moscow, Izd. Inostr. Lit., 1961.

19. GOLUBEV V.V., Lectures on the Analytical Theory of Differential Equations. Moscow–Leningrad, Gostekhizdat, 1950.

20. McCRACKEN D. and DORN W., Numerical Methods and FORTRAN programming. Moscow, MIR, 1977.

21. KELLER J.B., The theory of branching of solutions of ordinary differential equations. In: The Theory of Branching and Nonlinear Problems in Eigenvalues. Moscow, MIR, 1974.

Translated by J.J.D.